A FORMULAE FOR FIB (x-n) AND FIB (x+2n)

Interpolation formulae are usually used when finding the unknown values of function for intermediate values of the argument. In these cases one often has to consider function $y = f(x)$ specified by tabular values $y_k = f(x_k)$ for a set of equidistant points x_k, (k=0,1,2,3…) where

$$\Delta X_k = X_{k+1} - X_k = \text{const}$$

The finite differences of the sequence y_k are naturally defined by the relations

$$\begin{aligned} Hy_k &= \Delta y_k = y_{k+1} - y_k \\ H^2 y_k &= \Delta(\Delta y_k) = Hy_{k+1} - Hy_k \\ &\text{-------------------------- ---} \\ H^n y_k &= \Delta(\Delta^{n-1} y_k) = H^{n-1} y_k - H^{n-1} y_k \end{aligned}$$

From the first equation we get

$$y_{k+1} = y_k + Hy_k = (1+H) y_k$$

Whence we derive in succesion

$$\begin{aligned} y_{k+2} &= (1+H) y_{k+1} = (1+H)^2 y_k \\ y_{k+3} &= (1+H) y_{k+2} = (1+H)^2 y_{k+1} = (1+H)^3 y_k \\ &\text{---} \\ y_{k+n} &= (1+H) y_{k+n-1} = (1+H)^n y_k \end{aligned}$$

Then the symbol H may be regarded as an operator that associates the function

$$\mathrm{Hy(x)} = \mathrm{y(x+1)} - \mathrm{y(x)}$$

With the function

$$\mathrm{y} = \mathrm{f(x)}$$

it is easy to verify that

$$\mathrm{H} = e^{D} - 1$$

or

$$\mathrm{H} = \sum_{k=0}^{\infty} \frac{D^k}{k!}$$

Where the D is operator of differentation

$$Dy(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = \ln(1+H)\,y(x)$$

and where

$$H^n = (-1+e^D)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{kD}$$

Leonardo Fibonacci of Pisa was a mathematician in 13th century, Italy. By charting the population of rabbits, he discovered a number series from which one can derive the Golden Mean. Here’s the beginning of the sequence:

$$1,1,2,3,5,8,13,21,34,55...$$

Each number is the sum of the two proceeding numbers

$$F(n+2)=F(n+1)+F(n)$$

where

$$\begin{array}{ll} F(1)=1 & F(6)=8 \\ F(2)=1 & F(7)=13 \\ F(3)=2 & F(8)=21 \\ F(4)=3 & F(9)=34 \\ F(5)=5 & F(10)=55 \end{array}$$

Etc.

If you prefer operator of the finite differences in this formula, then here is another form

$$(1+H)^2F(n)=(1+H)F(n)+F(n)$$

or

$$(1+H)^2=(1+H)+1$$

It may be noted that it is possible to derive operator's formulas for Fibonacci's numbers. As will be seen , we usually deal with this operator's equations for Fibonacci's series

$$H=\frac{1}{1+H}$$

and

$$1+H=\frac{1}{H}$$

We get the formula for the n th degree of the operator H as

$$H^{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+H)^{k}$$

Setting and replacing operator

$$H^{n}$$

by

$$H^{n}=\left(\frac{1}{1+H}\right)^{n}$$

we get

$$\left(\frac{1}{1+H}\right)^{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+H)^{k}$$

Then for Fibonacci's series we have

$$F(x-n)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F(x+k)$$

The simplest examples show that, generally , this equation possesses an infinite number of solutions

$$F(x-1) = F(x+1) - F(x)$$
$$F(x-2) = F(x+2) - 2F(x+1) + F(x)$$
$$F(x-3) = F(x+3) - 3F(x+2) + 3F(x+1) - F(x)$$
$$F(x-4) = F(x+4) - 4F(x+3) + 6F(x+2) - 4F(x+1) + F(x)$$
$$F(x-5) = F(x+5) - 5F(x+4) + 10F(x+3) - 10F(x+2) + 5F(x+1) - F(x)$$

$$- -$$

Etc.

It is obvious from the structure of this system that all the coefficients are the numbers of the Pascal's triangle.

Let us determine the Fibonacci's number of the form

$$F(x+2n)$$

Substituing operator's equation for the Fibonacci's series

$$H^k = \left(\frac{1}{1+H)}\right)^k$$

in the equation

$$(1+H)^n = \sum_{k=0}^{n} \binom{n}{k} H^k$$

we have

$$(1+H)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(1+H)^k}$$

or

$$(1+H)^n = \frac{\sum_{k=0}^{n}\binom{n}{k}(1+H)^{n-k}}{(1+H)^n}$$

now we get

$$(1+H)^{2n} = \sum_{k=0}^{n}\binom{n}{k}(1+H)^{n-k}$$

Thus, this solution is written in the form containing Fibonacci's numbers

$$F(x+2n) = \sum_{k=0}^{n}\binom{n}{k}F(x+n-k)$$

Looking generally, this equation possesses an infinity number of solutions

$$F(x+2) = F(x+1) + F(x)$$
$$F(x+4) = F(x+2) + 2F(x+1) + F(x)$$
$$F(x+6) = F(x+3) + 3F(x+2) + 3F(x+1) + F(x)$$
$$F(x+8) = F(x+4) + 4F(x+3) + 6F(x+2) + 4F(x+1) + F(x)$$
$$F(x+10) = F(x+5) + 5F(x+4) + 10F(x+3) + 10F(x+2) + 5F(x+1) + F(x)$$

Etc…

We obtain an infinite system of equations which coefficients are the numbers of Pascal's triangle.